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A Unified Approach for Nondifferentiable Functions

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Let $\psi(x)$ denote the distance between x and the nearest integer, and fix $0<a<1$, $ab>1$, where b is not necessarily an integer. Then for any sequence θ_n of phases, the function $f(x)=\sum_{n=0}^{\infty} a^n\psi(b^n x+\theta_n)$ has no right (left) derivative at any point x and $2+(\log a/\log b)$ is the box-counting dimension of the graph of f. The crucial step is to obtain the smallest Lipschitz class to which f belongs. © 1994 Academic Press, Inc.

1. INTRODUCTION

In 1918, Knopp [11] gave an example of a continuous function without a derivative at any point. He defined the function as $K(x)=\sum_{n=0}^{\infty} a^n\psi(b^n x)$, where $\psi(x)$ denotes the distance between x and the nearest integer. The conditions on the parameters a and b for the nonexistence of the derivative of this function as given by Knopp are $0<a<1$, b is a positive even integer, and $ab>4$. Prior to Knopp, and even later, several mathematicians studied this functions under the restriction that b is an integer. For the case $b=2$ and $a=\frac{1}{2}$ see, e.g., Takagi [14], Hildebrand [8], and de Rham [3]; van der Waerden [16] considered the case $b=10$ and $a=\frac{1}{10}$.

In this work, we study the function $K(x)$ under the weaker conditions that $0<a<1$ and $ab>1$ where b is not necessarily an integer. Let us define the following two classes of function, the first of which is well known.

- A function f belongs to the *Lipschitz class of order* α (Lip(α)), if there exists a constant C such that $|f(x)-f(y)|\leqslant C\,|x-y|^{\alpha}$ for any x and y such that $|x-y|\leqslant 1$.
- A function f belongs to the class Inc(α), if there is a constant $c>0$ such that for each $x\in\mathbb{R}$, and for any $h\in(0,1)$ there is a $t\in(x,x+h)$ such that $|f(t)-f(x)|>ch^{\alpha}$. (Remark: Inc(α) stands for increments of order α).

The main result is that K belongs to each of these two classes with $\alpha=-\log a/\log b$. Under the same conditions (i.e., $0<a<1$, $ab>1$), for

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any sequence θ_n of phases, the function $f(x)=\sum_{n=0}^{\infty} a^n\psi(b^n x+\theta_n)$ belongs to $\mathrm{Lip}(\alpha)\cap \mathrm{Inc}(\alpha)$. As a consequence of this result, together with the fact that the box-counting dimension of the graph of f is $2-\alpha$, the nondifferentiability of $f(x)$ is established.

The family of functions $\sum_{n=0}^{\infty} a^n\Phi(b^n x+\theta_n)$, where Φ is periodic and θ_n is an arbitrary sequence of phases, was introduced by Mauldin and Williams [13]. Further, the Ursell–Besicovitch function also belongs to this family.

2. Lipschitz Conditions

In this section, we study the Lipschitz conditions for functions of the form $f(x)=\sum_{n=}^{\infty} a^n f_n(b^n x)$, where f_n is a sequence of functions defined on the real line.

Lemma 2.1. *Let $0<a<1$, $b>1/a$, and $\alpha=-\log a/\log b$. Let us assume that f_n is a sequence of functions for which*

- *there is a constant L such that for all x, y, $|f_n(x)-f_n(y)|\leqslant L\,|x-y|$,*
- *there is a constant D such that for all x, y, $|f_n(x)-f_n(y)|\leqslant D$, then $f(x)=\sum_{n=0}^{\infty} a^n f_n(b^n x)$ belongs to* $\mathrm{Lip}(\alpha)$.

Proof. Let m be an integer. If $b^{-m-1}\leqslant |x-y|\leqslant b^{-m}$, then

$$|f(x)-f(y)|\leqslant L\sum_{n=0}^{m-1} a^n b^{n-m}+D\sum_{n=m}^{\infty} a^n\leqslant\left(\frac{L}{ab-1}+\frac{D}{1-a}\right)a^m;$$

since $b^{-\alpha}=a$, with the constant $C=b^{\alpha}\{L/(ab-1)+D/(1-a)\}$, the inequality $|f(x)-f(y)|\leqslant C\,|x-y|^{\alpha}$ is fulfilled. ∎

Corollary 2.2. *If $0<a<1$, $ab>1$, $\alpha=-\log a/\log b$, and θ_n is any sequence of phases, then the function $f(x)=\sum_{n=0}^{\infty} a^n\psi(b^n x+\theta_n)$ belongs to* $\mathrm{Lip}(\alpha)$.

3. Order of the Increments with Harmonic Analysis

This is the central section: we prove that any function $f(x)=\sum_{n=0}^{\infty} a^n\psi(b^n x+\theta_n)$ belongs to $\mathrm{Inc}(\alpha)$. The increments $f(t)-f(x)$ are analyzed through the integrals

$$\frac{1}{Nb^{-m}}\int_x^{x+Nb^{-m}} (f(t)-f(x))\cos 2\pi(b^m t+\theta_m)\,dt.$$

In order to study these quantities, a family of elementary integrals is introduced in the following:

$$J_k(x, \theta, N) = \frac{a^k}{Nb^k} \int_x^{x+Nb^k} \psi(t) \cos 2\pi(b^{-k}t + \theta)\, dt.$$

Five preliminary lemmas are required to carry on the discussion.

LEMMA 3.1. *$|J_k(x, \theta, N)|$ is bounded by $a^k/2$ if $k \geq 0$ and by $(ab)^k/\pi$ if $k < 0$.*

Proof. If $k \geq 0$, then $|J_k(x, \theta, N)| \leq a^k/2$ since $|\psi(x)| \leq \frac{1}{2}$. If $k < 0$, integrating by parts, we obtain

$$J_k(x, \theta, N) = \frac{a^k}{2\pi N} \left\{ [\psi(x + Nb^k) - \psi(x)] \sin 2\pi(b^{-k}x + \theta) - \int_x^{x+Nb^k} \psi'(t) \sin 2\pi(b^{-k}t + \theta)\, dt \right\}.$$

Since $|\psi'(x)| \leq 1$, one has $|J_k(x, \theta, N)| \leq (ab)^k/\pi$. ∎

LEMMA 3.2. *For all x, y, for any integer N, $|\sum_{n=0}^N \cos(x + ny)| \leq 1/|\sin(y/2)|$.*

Proof. If $w = \sum_{n=0}^N e^{i(x+ny)}$, then $w = e^{ix}(1 - e^{i(N+1)y})/(1 - e^{iy})$. It follows that $|w| \leq 2/|1 - e^{iy}| = 1/|\sin(y/2)|$. As $\sum_{n=0}^N \cos(x + ny)$ is the real part of w, the required inequality is true. ∎

Remark. The previous argument is found in the textbook by Valiron [15, p. 13].

LEMMA 3.3. *If b^{-k} is an integer m, then*

$$\lim_{N \to \infty} J_k(x, \theta, N) = -\frac{a^k\{1 - (-1)^m\} \cos 2\pi\theta}{2\pi^2 m^2}$$

uniformly in x and in θ. If b^{-k} is not an integer, then $\lim_{N \to \infty} J_k(x, \theta, N) = 0$ uniformly in x and in θ.

Proof. If b^{-k} is an integer m, then

$$J_k(x, \theta, N) = \frac{ma^k}{N} \int_x^{x+N/m} \Psi(t) \cos 2\pi(mt + \theta)\, dt$$

$$\lim_{N \to \infty} J_k(x, \theta, N) = a^k \int_0^1 \psi(t) \cos 2\pi(mt + \theta)\, dt$$

$$= -\frac{a^k\{1 - (-1)^m\} \cos 2\pi\theta}{2\pi^2 m^2}.$$

The convergence is uniform in x. If b^{-k} is not an integer, then

$$J_k(x, \theta, N) = \frac{a^k}{Nb^k} \int_x^{x+Nb^k} \psi(t) \cos 2\pi(b^{-k}t + \theta)\, dt$$

$$J_k(x, \theta, N) = \frac{a^k}{Nb^k} \int_x^{x+1} \psi(t) \sum_{L=0}^{L(t)} \cos 2\pi(b^{-k}(t+L) + \theta)\, dt,$$

where $L(t)$ is the largest integer less than or equal to $x - t + Nb^k$.

From the previous lemma, it follows that

$$|J_k(x, \theta, N)| \leqslant \frac{a^k}{Nb^k} \int_x^{x+1} \psi(t)/|\sin(\pi b^{-k})|\, dt \leqslant \frac{a^k}{2Nb^k\, |\sin(\pi b^{-k})|}.$$

Therefore $\lim_{N \to \infty} J_k(x, \theta, N) = 0$ uniformly in x and in θ. ∎

LEMMA 3.4. *If b is a rational number greater than one and if b^k is an integer, where k is a positive integer, then b is an integer.*

Proof. Let $b = p/q$, where p and q are relatively prime. Then $p^k = b^k q^k$. Thus any prime factor of q must also be a prime factor of p, and this conflicts with the fact that p and q are relatively prime. So $q = 1$ and b is an integer. ∎

LEMMA 3.5. *Let b be a real number greater than one, and let us assume that the set $S = \{k \in \mathbb{N} : b^k \in \mathbb{N}\} \neq \{0\}$. Then there is an integer r such that $S = \{0, r, 2r, 3r, ...\}$.*

Proof. We define r as the least positive integer such that b^r is an integer. Let us assume that $S \neq \{0, r, 2r, 3r, ...\}$. So there is an integer for which $n \in S$ and n is not integral multiple of r. Hence $n = rq + h$, where q and h are integers and $0 < h < r$. Thus $b^h = b^n/(b^r)^q$ is rational. So $(b^h)^r = (b^r)^h$ is an integer. By Lemma 3.4, b^h is an integer. This is a contradiction. ∎

THEOREM 3.6. *If $0 < a < 1$, $ab > 1$, $\alpha = -\log a/\log b$, and if θ_n is any sequence of phases, then the function $f(x) = \sum_{n=0}^{\infty} a^n \psi(b^n x + \theta_n)$ belongs to* $\mathrm{Inc}(\alpha)$.

Proof. We consider the quantity

$$\frac{1}{Nb^{-m}} \int_x^{x+Nb^{-m}} f(t) \cos 2\pi(b^m t + \theta_m)\, dt = a^m \sum_{k=-m}^{\infty} J_k(x_k, \theta'_k, N),$$

where $x_k = b^{m+k}x + \theta_{m+k}$ and $\theta'_k = \theta_m - b^{-k}\theta_{m+k}$.

The key to the proof is showing that if N is sufficiently large, then $\sum_{k \neq 0} |J_k(x_k, \theta'_k, N)| < |J_0(x, 0, N)|$.

First remark. $\forall x \in \mathbb{R}$, $\forall N \in \mathbb{N}$, $J_0(x, 0, N) = -1/\pi^2$.

Second remark (Lemma 3.1). It is possible to choose an integer M such that for any sequence x_k, $\forall N \in \mathbb{N}$, $\sum_{|k| > M} |J_k(x_k, \theta'_k, N)| < 0.01$.

Third remark. Case 1. No integral power of b belongs to $\mathbb{N}$.

According to Lemma 3.3, it is possible to choose an integer N such that for all $x, \theta \in \mathbb{R}$, for any $k \in [-M, M]$, one has $|J_k(x, \theta, N)| < 0.01/M$.

So for any $m \geqslant 0$, $\sum_{k \neq 0} |J_k(x_k, \theta'_k, N)| < 0.03 < 1/\pi^2 = |J_0(x, 0, N)|$ and

$$\frac{1}{Nb^{-m}} \int_x^{x + Nb^{-m}} f(t) \cos 2\pi(b^m t + \theta_m)\, dt < -0.05 a^m.$$

Case 2. At least one integral power of b belongs to $\mathbb{N}$.

According to Lemma 3.5, $\{k \in \mathbb{N} : b^k \in \mathbb{N}\} = \{0, r, 2r, 3r, ...\}$. It is possible (Lemma 3.3) to choose an integer N such that for all $x, \theta \in \mathbb{R}$, for any $k \in [1, M]$, one has $|J_k(x, \theta, N)| < \eta/M$,

$$|J_{k}(x, \theta, N) < \eta/M \qquad \text{if} \quad k \not\equiv 0 \bmod r$$

and

$$|J_{-k}(x, \theta, N)| < \frac{a^{-k}}{\pi^2 b^{2k}} + \eta/M \qquad \text{if} \quad k \equiv 0 \bmod r$$

(Lemma 3.3). Since $ab > 1$, then

$$\frac{a^{-k}}{\pi^2 b^{2k}} \leqslant \frac{1}{ab\pi^2 b^k}$$

if k is positive.

So for any $m \geqslant 0$,

$$\sum_{k \neq 0} |J_k(x_k, \theta'_k, N)| < \frac{1}{ab\pi^2(b^r - 1)} + 3\eta < \frac{1}{\pi^2}$$

if η is sufficiently small.

In both cases, it is possible to find a positive number ε such that

$$\frac{1}{Nb^{-m}} \int_x^{x + Nb^{-m}} f(t) \cos 2\pi(b^m t + \theta_m)\, dt < -\varepsilon a^m \qquad \text{for} \quad m = 0, 1, 2, ...$$

But

$$\int_x^{x+Nb^{-m}} f(t)\cos 2\pi(b^m t+\theta)\,dt$$

$$=\int_x^{x+Nb^{-m}} [f(t)-f(x)]\cos 2\pi(b^m t+\theta)\,dt.$$

From the triangle inequality for integrals, we deduce that there exists at least one t such that $x<t<x+Nb^{-m}$ and $|f(t)-f(x)|>\varepsilon a^m$.

Given $h>0$, we can find an integer m such that $Nb^{-m}<h<bNb^{-m}$. Using the identity $1/(b^m)^\alpha=a^m$ and the last inequality, we obtain $|f(t)-f(x)|>\varepsilon(b^{-m})^\alpha>\varepsilon(h/(bN))^\alpha$. For $c=\varepsilon(1/bN)^\alpha$, the theorem is proved. ∎

COROLLARY 3.7. *If $0<a<1$, $ab>1$, and θ_n is any sequence of phases, then the function $\sum_{n=0}^{\infty} a^n\psi(b^n x+\theta_n)$ has no right (left) derivative at any point x.*

Indeed, if $\alpha\in(0,1)$ and $f\in\mathrm{Inc}(\alpha)$, then at every point there is no right derivative. In fact, if x is a given point on the real axis, there is a sequence x_n such that $x_n>x$ and $|f(x_n)-f(x)|>c\,|x_n-x|^\alpha$; so the differential quotients $(f(x_n)-f(x))/(x_n-x)$ are not bounded and do not converge. There is no right derivative. There is also no left derivative at any point since ψ is an even function. ∎

4. THE DIMENSION OF THE GRAPH OF NONDIFFERENTIABLE FUNCTIONS

In this section, the box-counting dimension of the previous nondifferentiable functions is computed. We recall the notion of dimension introduced for the first time by Bouligand [2] and studied by Falconer [6]. In order to define the dimension of a compact subset E of the plane, a family of squares is defined; if h is a positive number, if i and j are two integers, $Q_{i,j}(h)$ is the square $[ih, ih+h]\times[jh, jh+h]$; $N_E(h)$ is the cardinality of $\{(i,j):Q_{i,j}(h)\cap E\neq\varnothing\}$; the *box-counting dimension* is the number $\dim(\mathrm{E})=\inf\{\alpha:N_E(h)\text{ is }O(h^{-\alpha})\}$.

THEOREM 4.1. *If f is defined on $[0,1]$, if $\alpha\in(0,1)$, and if $f\in\mathrm{Lip}(\alpha)\cap\mathrm{Inc}(\alpha)$, then the box-counting dimension of the graph of f is $2-\alpha$.*

This theorem is Corollary 11.2 in Falconer's book [6].

COROLLARY 4.2. *If* $0<\alpha<1$, $ab>1$, *and* θ_n *is any sequence of phases, then the box-counting dimension of the graph of the function* $\sum_{n=0}^{\infty} a^n\psi(b^n x+\theta_n)$ *is* $2+(\log a/\log b)$.

Comments.

- A slightly different situation has been investigated by B. Dubuc. In [4], he determined the box-counting dimension of the binary random curves defined by $f(x)=\sum_{n=0}^{\infty} a^n\psi(2^n x)\,\sigma_n(2^n x)$, where $\sigma_n(x)=\sum_{i=0}^{2^n-1} X_{i,n}\,\delta_i(x)$, each $X_{i,n}$ is a binary random variable taking a value in $\{+1,-1\}$ with equal probability, and $\delta_i(x)=1$ for x in $[i,i+1]$ and 0 otherwise. This dimension $\dim(G_f)$ is $\log(4a)/\log 2$.
- We mention a theorem due to Kaplan *et al.* [10]. If $0<a<1$, $b>1/a$, and $q:\mathbb{R}\to\mathbb{R}$ is C^1 and almost periodic, then the box-counting dimension of the graph G_h of the function $h(x)=\sum_{n=0}^{\infty} a^n q(b^n x)$ is $\dim(G_h)=2+(\log a/\log b)$.

5. CONSEQUENCES FOR THE URSELL–BESICOVITCH FUNCTION

In Hobson [9], another nondifferentiable function is described. This function is $B(x)=\sum_{n=0}^{\infty} a^n\phi(b^n x)$, where $\phi(x)=x$ for $x\in[-\frac{1}{2},\frac{1}{2}]$, $\phi(x)=1-x$ for $x\in[\frac{1}{2},\frac{3}{2}]$, and $\phi(x)=\phi(x+2)$ for all x in $\mathbb{R}$. The function $B(x)$ is called the Ursell–Besicovitch function by Edgar [5].

THEOREM 5.1. *If* $0<a<1$, $ab>1$, *then the Ursell–Besicovitch function* $B(x)$ *belongs to* $\operatorname{Lip}(\alpha)\cap\operatorname{Inc}(\alpha)$ *and has no right (left) derivative at any point* x. *The box-counting dimension of the graph of* B *is* $2+(\log a/\log b)$.

Proof. There is a simple identity:

$$B(x)=\sum_{n=0}^{\infty} 2a^n\left\{\psi\left(2b^n x+\frac{1}{4}\right)-\frac{1}{4}\right\}$$

$$=\left\{\sum_{n=0}^{\infty} 2a^n\psi\left(2b^n x+\frac{1}{4}\right)\right\}-\frac{1}{2(1-a)}.$$

The theorem is a consequence of Theorems 3.6 and 4.1. ∎

6. CONCLUSION

We add some remarks on the Weierstrass function. Let $0<a<1$, $ab>1$, and $\alpha=-\log a/\log b$. The Weierstrass function

$$W(x)=\sum_{n=0}^{\infty} a^n\cos(\pi b^n x),$$

or, more generally,

$$\sum_{n=0}^{\infty} a^n \cos(\pi b^n x - \theta_n),$$

belongs to $\mathrm{Lip}(\alpha) \cap \mathrm{Inc}(\alpha)$ as was proved by the authors in [1]. According to Section 4, the dimension of the graph of W is equal to $2-\alpha$. To our knowledge, the first proof of this fact was given by Kaplan *et al.* [10].

The Weierstrass function $W(x)$, the Knopp function, and the Ursell–Besicovitch function make use of parameters a and b. If $a>0$, $ab>1$, then each of these functions are nondifferentiable: this is a negative statement. A much more positive statement is proved in this paper. If $\alpha = -\log a/\log b$, then $\mathrm{Lip}(\alpha)$ is the smallest Lipschitz class to which each of these functions belongs.

A situation which was not considered in this paper is the critical case $ab=1$. This situation is much more difficult to study. If $b>1$, what can be said about the increments of $\sum_{n=0}^{\infty} \psi(b^n x + \theta_n)/b^n$? One of the referees has reported that very strong results have been shown by Kôno [12] in this direction. *If* $T(x) = \sum_{n=0}^{\infty} \psi(2^n x)/2^n$ is the Takagi function, then, almost everywhere in x,

$$\limsup_{h\to 0} \frac{T(x+h)-T(x)}{h\sqrt{\log(1/|h| \log\log\log(1/|h|)}} = \sqrt{\frac{2}{\log 2}}$$

and

$$\liminf_{h\to 0} \frac{T(x+h)-T(x)}{h\sqrt{\log(1/|h| \log\log\log(1/|h|)}} = -\sqrt{\frac{2}{\log 2}}$$

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